

One-shot entanglement-assisted quantum and classical communication

Nilanjana Datta and Min-Hsiu Hsieh

Abstract—We study entanglement-assisted quantum and classical communication over a single use of a quantum channel, which itself can correspond to a finite number of uses of a channel with arbitrarily correlated noise. We obtain characterizations of the corresponding one-shot capacities by establishing upper and lower bounds on them in terms of the difference of two smoothed entropic quantities. In the case of a memoryless channel, the upper and lower bounds converge to the known single-letter formulas for the corresponding capacities, in the limit of asymptotically many uses of it. Our results imply that the difference of two smoothed entropic quantities characterizing the one-shot entanglement-assisted capacities serves as a one-shot analogue of the mutual information, since it reduces to the mutual information, between the output of the channel and a system purifying its input, in the asymptotic, memoryless scenario.

I. INTRODUCTION

An important class of problems in quantum information theory concerns the evaluation of information transmission capacities of a quantum channel. The first major breakthrough in this area was made by Holevo [1], [2], and Schumacher and Westmoreland [3], who obtained an expression for the capacity of a quantum channel for transmission of classical information. They proved that this capacity is characterized by the so-called “Holevo quantity”. Expressions for various other capacities of a quantum channel were obtained subsequently, the most important of them perhaps being the capacity for transmission of quantum information. It was established by Lloyd [4], Shor [5], and Devetak [6]. Both of the above capacities require regularization over asymptotically many uses of the channel. The classical capacity formula is given in terms of a regularized Holevo quantity while the quantum capacity formula is given in terms of a regularized coherent information.

Even though these regularized expressions are elegant, they are not useful because the regularization prevents one from explicitly computing the capacity of any given channel. Moreover, since these capacities are in general not additive, surprising effects like superactivation can occur [7], [8]. Hence it is more desirable to obtain expressions for capacities which are given in terms of *single-letter* formulas, and therefore have the attractive feature of being exempt from regularization.

The quantum and classical capacities of a quantum channel can be increased if the sender and receiver share entangled states, which they may use in the communication protocol. From superdense coding we know that the classical capacity

of a noiseless quantum channel is exactly doubled in the presence of a prior shared maximally entangled state (an ebit). For a noisy quantum channel too, access to an entanglement resource can lead to an enhancement of both its classical and quantum capacities. The maximum asymptotic rate of reliable transmission of classical (quantum) information through a quantum channel, in the presence of unlimited prior shared entanglement between the sender and the receiver, is referred to as the entanglement-assisted classical (quantum) capacity of the channel.

Quantum teleportation and superdense coding together imply the following simple relation between quantum and classical communication through a noiseless qubit channel: if the sender and receiver initially share an ebit of entanglement, then transmission of one qubit is equivalent to transmission of two classical bits¹. This relation carries over to the asymptotic setting under the assumption of unlimited prior shared entanglement between sender and receiver [13]. One can then show that the entanglement-assisted quantum capacity of a quantum channel is equal to half of its entanglement-assisted classical capacity. In fact, the entanglement-assisted classical capacity was the first capacity for which a single-letter formula was obtained. This result is attributed to Bennett, Shor, Smolin and Thapliyal [9]. Its proof was later simplified by Holevo [10], and an alternative proof was given in [11]. A trade-off formula for the entanglement-assisted quantum capacity region was obtained by Devetak, Harrow, and Winter [12], [13].

The different capacities of a quantum channel were originally evaluated in the so-called *asymptotic, memoryless scenario*, that is, in the limit of asymptotically many uses of the channel, under the assumption that the channel was memoryless (i.e., there is no correlation in the noise acting on successive inputs to the channel). In reality however, the assumption of channels being memoryless, and the consideration of an asymptotic scenario is not necessarily justified. A more fundamental and practical theory of information transmission through quantum channels is obtained instead in the so-called *one-shot scenario* (see e.g. [14] and references therein) in which channels are available for a finite number of uses, there is a correlation between their successive actions, and information transmission can only be achieved with finite accuracy. The optimal rate at which information can be transmitted through a single use of a quantum channel (up to a given accuracy) is called its one-shot capacity. Note that a single use of the channel can itself correspond to a finite number

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¹Without prior shared entanglement, one can only send one classical bit through a single use of a noiseless qubit channel. Moreover, a noiseless classical bit channel cannot be used to transmit a qubit.

of uses of a channel with arbitrarily correlated noise. The one-shot capacity of a classical-quantum channel was studied in [15], [16], [17], whereas the one-shot capacity of a quantum channel for transmission of quantum information was evaluated in [14], [18].

The fact that the one-shot scenario is more general than the asymptotic, memoryless one is further evident from the fact that the asymptotic capacities of a memoryless channel can directly be obtained from the corresponding one-shot capacities. Moreover, one-shot capacities also yield the asymptotic capacities of channels with memory (see e.g. [18]). In [19] the asymptotic entanglement-assisted classical capacity of a particular class of quantum channels with long-term memory, given by convex combinations of memoryless channels, was evaluated. The classical capacity of this channel was obtained in [20]. A host of results on asymptotic capacities of channels with memory can be attributed to Bjelakovic et al (see [21], [22] and references therein).

In this paper we study the one-shot entanglement-assisted quantum and classical capacities of a quantum channel. The requirement of finite accuracy is implemented by imposing the constraint that the error in achieving perfect information transmission is at most ε , for a given $\varepsilon > 0$. We completely characterize these capacities by deriving upper and lower bounds for them in terms of the same smoothed entropic quantities.

Our lower and upper bounds on the one-shot entanglement-assisted quantum and classical capacities converge to the known single-letter formulas for the corresponding capacities in the asymptotic, memoryless scenario. Our results imply that the difference of two smoothed entropic quantities characterizing these one-shot capacities serves as a one-shot analogue of the mutual information, since it reduces to the mutual information between the output of a channel and a system purifying its input, in the asymptotic, memoryless setting.

The paper is organized as follows. We begin with some notations and definitions of various one-shot entropic quantities in Section II. In Section III, we first introduce the one-shot entanglement-assisted quantum communication protocol and then give upper and lower bounds on the corresponding capacity in Theorem 8. The proof of the upper bound is given in the same section, whereas the proof of the lower bound is given in Appendix B. The case of one-shot entanglement-assisted classical communication is considered in Section IV, and the bounds on the corresponding capacity is given in Theorem 13, the proof of which is given in the same section. In Section V, we show how our results in the one-shot setting can be used to recover the known single-letter formulas in the asymptotic, memoryless scenario. Finally, we conclude in Section VI.

II. NOTATIONS AND DEFINITIONS

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of linear operators acting on a finite-dimensional Hilbert space \mathcal{H} , and let $\mathcal{D}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ be the set of positive operators of unit trace (states):

$$\mathcal{D}(\mathcal{H}) = \{\rho \in \mathcal{B}(\mathcal{H}) : \text{Tr } \rho = 1\}.$$

Furthermore, let

$$\mathcal{D}_{\leq}(\mathcal{H}) := \{\rho \in \mathcal{B}(\mathcal{H}) : \text{Tr } \rho \leq 1\}.$$

Throughout this paper, we restrict our considerations to finite-dimensional Hilbert spaces and denote the dimension of a Hilbert space \mathcal{H}_A by $|A|$.

For any given pure state $|\psi\rangle \in \mathcal{H}$, we denote the projector $|\psi\rangle\langle\psi|$ simply as ψ . For an operator $\omega^{AB} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, let $\omega^A := \text{Tr}_B \omega^{AB}$ denote its restriction to the subsystem A . For given orthonormal bases $\{|i^A\rangle\}_{i=1}^d$ and $\{|i^B\rangle\}_{i=1}^d$ in isomorphic Hilbert spaces $\mathcal{H}_A \simeq \mathcal{H}_B \simeq \mathcal{H}$ of dimension d , we define a maximally entangled state (MES) of Schmidt rank d to be

$$|\Phi\rangle^{AB} = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i^A\rangle \otimes |i^B\rangle. \quad (1)$$

Let \mathbb{I}_A denote the identity operator in $\mathcal{B}(\mathcal{H}_A)$, and let $\tau^A := \mathbb{I}_A/|A|$ denote the completely mixed state in $\mathcal{D}(\mathcal{H}_A)$.

In the following we denote a completely positive trace-preserving (CPTP) map $\mathcal{E} : \mathcal{B}(\mathcal{H}_A) \mapsto \mathcal{B}(\mathcal{H}_B)$ simply as $\mathcal{E}^{A \rightarrow B}$, and denote the identity map as id . Similarly, we denote an isometry $U : \mathcal{H}_A \mapsto \mathcal{H}_B \otimes \mathcal{H}_C$ simply as $U^{A \rightarrow BC}$.

The trace distance between two operators A and B is given by

$$\|A - B\|_1 := \text{Tr}[\{A \geq B\}(A - B)] - \text{Tr}[\{A < B\}(A - B)],$$

where $\{A \geq B\}$ denotes the projector onto the subspace where the operator $(A - B)$ is non-negative, and $\{A < B\} := \mathbb{I} - \{A \geq B\}$. The fidelity of two states ρ and σ is defined as

$$F(\rho, \sigma) := \text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}} = \|\sqrt{\rho} \sqrt{\sigma}\|_1. \quad (2)$$

Note that the definition of fidelity can be naturally extended to subnormalized states. The trace distance between two states ρ and σ is related to the fidelity $F(\rho, \sigma)$ as follows (see e. g. [23]):

$$1 - F(\rho, \sigma) \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F^2(\rho, \sigma)}. \quad (3)$$

The *entanglement fidelity* of a state $\rho^Q \in \mathcal{D}(\mathcal{H}_Q)$, with purification $|\Psi\rangle^{RQ}$, with respect to a CPTP map $\mathcal{A} : \mathcal{B}(\mathcal{H}_Q) \mapsto \mathcal{B}(\mathcal{H}_Q)$ is defined as

$$F_e(\rho^Q, \mathcal{A}) := \langle \Psi^{QR} | (\text{id}^R \otimes \mathcal{A}) (|\Psi\rangle\langle\Psi|^{RQ}) | \Psi^{RQ} \rangle. \quad (4)$$

We will also make use of the following fidelity criteria. The *minimum fidelity* of a map $\mathcal{T} : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ is defined as

$$F_{\min}(\mathcal{T}) := \min_{|\phi\rangle \in \mathcal{H}} \langle \phi | \mathcal{T}(|\phi\rangle\langle\phi|) | \phi \rangle. \quad (5)$$

The *average fidelity* of a map $\mathcal{T} : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ is defined as

$$F_{\text{av}}(\mathcal{T}) := \int d\phi \langle \phi | \mathcal{T}(|\phi\rangle\langle\phi|) | \phi \rangle. \quad (6)$$

The results in this paper involve various entropic quantities. The von Neumann entropy of a state $\rho^A \in \mathcal{D}(\mathcal{H}_A)$ is given by $H(A)_\rho = -\text{Tr } \rho^A \log \rho^A$. Throughout this paper we take the logarithm to base 2. For any state $\rho^{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ the quantum mutual information is defined as

$$I(A : B)_\rho := H(A)_\rho + H(B)_\rho - H(AB)_\rho. \quad (7)$$

The following generalized relative entropy quantity, referred to as the max-relative entropy, was introduced in [24]:

Definition 1: The max-relative entropy of two operators $\rho \in \mathcal{D}_{\leq}(\mathcal{H})$ and $\sigma \in \mathcal{B}(\mathcal{H})$ is defined as

$$D_{\max}(\rho||\sigma) := \log \min\{\lambda : \rho \leq \lambda\sigma\}. \quad (8)$$

We also use the following min- and max- entropies defined in [25], [26], [27]:

Definition 2: Let $\rho^{AB} \in \mathcal{D}_{\leq}(\mathcal{H}_A \otimes \mathcal{H}_B)$. The min-entropy of A conditioned on B is defined as

$$H_{\min}(A|B)_{\rho} = \max_{\sigma^B \in \mathcal{D}(\mathcal{H}_B)} [-D_{\max}(\rho^{AB}||\mathbb{I}_A \otimes \sigma^B)].$$

Definition 3: For any $\rho \in \mathcal{D}(\mathcal{H})$, we define the ε -ball around ρ as follows

$$\mathcal{B}^{\varepsilon}(\rho) = \{\bar{\rho} \in \mathcal{D}_{\leq}(\mathcal{H}) : F^2(\bar{\rho}, \rho) \geq 1 - \varepsilon^2\}.$$

Definition 4: Let $\varepsilon \geq 0$ and $\rho^{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. The ε -smoothed min-entropy of A conditioned on B is defined as

$$H_{\min}^{\varepsilon}(A|B)_{\rho} = \max_{\bar{\rho}^{AB} \in \mathcal{B}^{\varepsilon}(\rho^{AB})} H_{\min}(A|B)_{\bar{\rho}}.$$

The max-entropy is defined in terms of the min-entropy via the following duality relation [25], [26], [28]:

Definition 5: Let $\rho^{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and let $\rho^{ABC} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ be an arbitrary purification of ρ^{AB} . Then for any $\varepsilon \geq 0$

$$H_{\max}^{\varepsilon}(A|C)_{\rho} := -H_{\min}^{\varepsilon}(A|B)_{\rho}. \quad (9)$$

In particular, if ρ^{AB} is a pure state, then

$$H_{\min}^{\varepsilon}(A|B)_{\rho} = -H_{\max}^{\varepsilon}(A)_{\rho}. \quad (10)$$

For any state $\rho^{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, the smoothed max-entropy can be equivalently expressed as [25], [28]

$$H_{\max}^{\varepsilon}(A|B)_{\rho} := \min_{\bar{\rho}^{AB} \in \mathcal{B}^{\varepsilon}(\rho^{AB})} H_{\max}(A|B)_{\bar{\rho}}, \quad (11)$$

where

$$H_{\max}(A|B)_{\bar{\rho}} = \max_{\sigma^B \in \mathcal{D}(\mathcal{H}_B)} 2 \log F(\bar{\rho}^{AB}, \mathbb{I}_A \otimes \sigma^B). \quad (12)$$

Moreover, for any $\rho^A \in \mathcal{D}_{\leq}(\mathcal{H}_A)$,

$$H_{\max}(A)_{\rho} := 2 \log \text{Tr} \sqrt{\rho^A}. \quad (13)$$

Various properties of the entropies defined above, which we employ in our proofs, are given in Appendix A.

III. ONE SHOT ENTANGLEMENT-ASSISTED QUANTUM CAPACITY OF A QUANTUM CHANNEL

Entanglement-assisted quantum information transmission through a quantum channel is also referred to as the ‘‘father’’ protocol [12], [13]. The goal of this section is to analyse the one-shot version of this protocol. In order to do so, we first study the protocol of one-shot entanglement assisted entanglement transmission through a quantum channel, which is detailed below. We obtain bounds on its capacity in terms of smoothed entropic quantities, and then prove how these

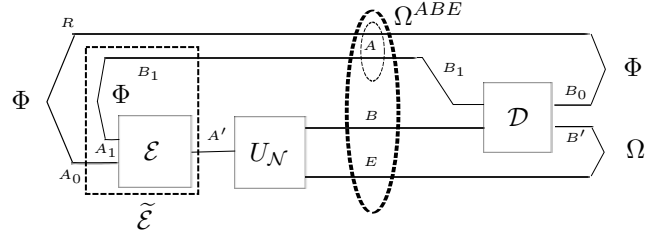


Fig. 1. One-shot entanglement-assisted entanglement transmission protocol. The task here is for Alice to transmit her half of the maximally entangled state A_0 , that she shares with an inaccessible reference R , to Bob with the help of a prior shared maximally entangled state $|\Phi\rangle^{A_1B_1}$ between her and Bob. In order to achieve this goal, Alice performs an encoding operation \mathcal{E} on systems A_0A_1 , and sends the resulting system A' through the channel \mathcal{N} (with Stinespring extension $U_{\mathcal{N}}$). Bob then performs his decoding operation \mathcal{D} on the channel output B and his half of the maximally entangled state B_1 so that finally the maximally entangled state $|\Phi\rangle^{RA_0}$ is shared between Bob and the reference R . The dashed rectangle denotes the CPTP map $\tilde{\mathcal{E}}$ defined through equation (16).

bounds readily yield bounds on the capacity of the one-shot ‘‘father’’ protocol.

The one-shot ε -error entanglement-assisted *entanglement transmission* protocol is as follows (see Fig. 1). The goal is for Alice to transmit half of a maximally entangled state, $|\Phi\rangle^{RA_0}$, that she shares with a reference R , to Bob through a quantum channel \mathcal{N} , with the help of a maximally entangled state $|\Phi\rangle^{A_1B_1}$ which she initially shares with him, such that finally the maximally entangled state $|\Phi\rangle^{RA_0}$ is shared between Bob and the reference R . We denote the latter as Φ^{RB_0} to signify that Alice’s system A_0 has been transferred to Bob. Note that $\log |A_1|$ denotes the number of ebits of entanglement consumed in the protocol and $\log |A_0|$ denotes the number of qubits transmitted from Alice to Bob. We require that Alice achieves her goal up to an accuracy ε , for some fixed $0 < \varepsilon < 1$.

Let Alice and Bob initially share a maximally entangled state $|\Phi\rangle^{A_1B_1}$. Without loss of generality, a one-shot ε -error entanglement-assisted entanglement transmission code of rate $r = \log |A_0|$ can then be defined by a pair of encoding and decoding operations $(\mathcal{E}, \mathcal{D})$ as follows:

- 1) Alice performs some encoding (CPTP map) $\mathcal{E}^{A_0A_1 \rightarrow A'}$. Let us denote the encoded state as

$$\xi^{B_1RA'} = (\text{id}^{B_1R} \otimes \mathcal{E}^{A_0A_1 \rightarrow A'}) (\Phi^{A_0R} \otimes \Phi^{A_1B_1}).$$

and denote the channel output state as

$$|\Omega\rangle^{B_1RBE_1} = (\mathbb{I}_{B_1R} \otimes U_{\mathcal{N}}^{A' \rightarrow BE}) |\xi\rangle^{B_1RA'E_1} \quad (14)$$

where $U_{\mathcal{N}}^{A' \rightarrow BE}$ is a Stinespring extension of the quantum channel $\mathcal{N}^{A' \rightarrow B}$ and $|\xi\rangle^{B_1RA'E_1}$ is some purification of the encoded state $\xi^{B_1RA'}$.

- 2) After receiving the channel output B , Bob performs a decoding operation $\mathcal{D}^{B_1B \rightarrow B_0B'}$ on the systems B_1, B in his possession. Denote the output state of Bob’s decoding operation by

$$\hat{\Omega}^{B_0B'REE_1} := (\text{id}^{REE_1} \otimes \mathcal{D}^{B_1B \rightarrow B_0B'}) (\Omega^{B_1RBE_1}). \quad (15)$$

For a quantum channel \mathcal{N} , and any fixed $0 < \varepsilon < 1$, a real number $r := \log |A_0|$ is said to be an ε -achievable rate if there

exists a pair $(\mathcal{E}, \mathcal{D})$ of encoding and decoding maps such that,

$$F_e(\tau^{A_0}, \tilde{\mathcal{D}} \circ \mathcal{N} \circ \tilde{\mathcal{E}}) = \langle \Phi^{RA_0} | \hat{\Omega}^{RB_0} | \Phi^{RA_0} \rangle \geq 1 - \varepsilon, \quad (16)$$

where $\tilde{\mathcal{D}} = \text{Tr}_{B'} \circ \mathcal{D}$, $\hat{\Omega}^{RB_0} = \text{Tr}_{B'E E_1} \hat{\Omega}^{B_0 B' R E E_1}$, and $\tilde{\mathcal{E}}^{A_0 \rightarrow A' B_1}$ is the CPTP map defined through the relation²

$$(\text{id}^R \otimes \tilde{\mathcal{E}})(\Phi^{A_0 R}) := (\text{id}^{RB_1} \otimes \mathcal{E})(\Phi^{A_0 R} \otimes \Phi^{A_1 B_1}).$$

Note that

$$\hat{\Omega}^{RB_0} := (\text{id}^R \otimes \tilde{\mathcal{D}} \circ \mathcal{N} \circ \tilde{\mathcal{E}})(\Phi^{RA_0}).$$

Definition 6 (Entanglement transmission fidelity): For any Hilbert space \mathcal{H}_{A_0} , the entanglement transmission fidelity of a quantum channel \mathcal{N} , in the presence of an assisting maximally entangled state, is defined as follows.

$$\mathbf{F}_e(\mathcal{N}, \mathcal{H}_{A_0}) := \max_{\tilde{\mathcal{E}}, \tilde{\mathcal{D}}} F_e(\tau^{A_0}, \tilde{\mathcal{D}} \circ \mathcal{N} \circ \tilde{\mathcal{E}}), \quad (17)$$

where τ^{A_0} denotes a completely mixed state in \mathcal{H}_{A_0} , and $\tilde{\mathcal{E}}, \tilde{\mathcal{D}}$ are CPTP maps.

Definition 7: Given a quantum channel $\mathcal{N}^{A' \rightarrow B}$ and a real number $0 < \varepsilon < 1$, the one-shot ε -error entanglement-assisted entanglement transmission capacity of \mathcal{N} is defined as follows:

$$E_{\text{ea}, \varepsilon}^{(1)}(\mathcal{N}) := \max\{\log |A_0| : \mathbf{F}_e(\mathcal{N}, \mathcal{H}_{A_0}) \geq 1 - \varepsilon\}.$$

Our main result of this section is the following theorem, which gives upper and lower bounds on $E_{\text{ea}, \varepsilon}^{(1)}(\mathcal{N})$ in terms of smoothed min- and max- entropies.

Theorem 8: For any fixed $0 < \varepsilon < 1$, $\kappa = 2\sqrt{2\sqrt{4\varepsilon}}$, and ε' being a positive number such that $\varepsilon = 2\sqrt{2\sqrt{27\varepsilon'} + 27\varepsilon'}$, the one-shot ε -error entanglement-assisted entanglement transmission capacity of a noisy quantum channel $\mathcal{N}^{A' \rightarrow B}$, in the case in which the assisting resource is a maximally entangled state, satisfies the following bounds:

$$\begin{aligned} \max_{\phi^{A'} \in \mathcal{D}(\mathcal{H}_{A'})} \frac{1}{2} \left[H_{\min}^{\varepsilon'}(A)_\psi - H_{\max}^{\varepsilon'}(A|B)_\psi \right] + 2 \log \varepsilon' \\ \leq E_{\text{ea}, \varepsilon}^{(1)}(\mathcal{N}) \leq \\ \max_{\phi^{A'} \in \mathcal{D}(\mathcal{H}_{A'})} \frac{1}{2} \left[H_{\min}^{\varepsilon}(A)_\psi - H_{\max}^{2\varepsilon+2\sqrt{\kappa}}(A|B)_\psi \right] + \log \frac{\sqrt{2}}{\varepsilon}, \end{aligned} \quad (18)$$

where the maximisation is over all possible inputs to the channel. In the above, ψ^{ABE} denotes the following state

$$|\psi\rangle^{ABE} := (\mathbb{I}_A \otimes U_{\mathcal{N}}^{A' \rightarrow BE})|\phi\rangle^{AA'} \quad (19)$$

where $U_{\mathcal{N}}^{A' \rightarrow BE}$ is a Stinespring isometry realizing the channel, and $|\phi\rangle^{AA'}$ denotes a purification of the input $\phi^{A'}$ to the channel.

Proof of the lower bound in (18). The proof is given in Appendix B.

Proof of the upper bound in (18). As stated in the beginning of Sec. III, any one shot ε -error entanglement-assisted entanglement transmission protocol (see Fig. 1) of a quantum channel \mathcal{N} consists of a pair $(\tilde{\mathcal{E}}, \tilde{\mathcal{D}})$ of encoding-decoding

maps such that the condition (16) holds. However, this condition along with Theorem 4 of [30] imply that there exists a partial isometry $V^{A_0 \rightarrow A'}$ such that

$$F_e(\tau^{A_0}, \mathcal{A} \circ V) \geq 1 - 2\varepsilon, \quad (20)$$

where $\mathcal{A} := \tilde{\mathcal{D}} \circ \mathcal{N}$, with $\mathcal{N} \equiv \mathcal{N}^{A' \rightarrow B}$ being the channel and $\tilde{\mathcal{D}}$ being the decoding map $\mathcal{D}^{B B_1 \rightarrow B_0 B'}$ followed by a partial trace over B' . The condition (20) in turn implies that

$$F(\hat{\Omega}^{RB_0}, \Phi^{RA_0}) \geq 1 - 2\varepsilon, \quad (21)$$

where

$$\hat{\Omega}^{RB_0} := (\text{id}^R \otimes \tilde{\mathcal{D}} \circ \mathcal{N} \circ V)(\Phi^{RA_0}).$$

By using Uhlmann's theorem [29], and the second inequality in (3), we infer from (21) that

$$\|\hat{\Omega}^{B_0 B' R E} - \Phi^{RB_0} \otimes \sigma^{B' E}\|_1 \leq 2\sqrt{2\sqrt{4\varepsilon}}, \quad (22)$$

for some state $\sigma^{B' E}$, where $\hat{\Omega}^{B_0 B' R E}$ denotes the output state of Bob's decoding operation, which in this case is defined by

$$\hat{\Omega}^{B_0 B' R E} := (\text{id}^{RE} \otimes \mathcal{D}^{B_1 B \rightarrow B_0 B'}) (\Omega^{B_1 R B E}), \quad (23)$$

with

$$|\Omega\rangle^{B_1 R B E} = (\mathbb{I}^{B_1 R} \otimes U_{\mathcal{N}}^{A' \rightarrow BE})|\xi\rangle^{RA' B_1} \quad (24)$$

where $|\xi\rangle^{RA' B_1}$ denotes the state resulting from the isometric encoding V on $|\Phi\rangle^{RA_0}$.

By the monotonicity of the trace distance under the partial trace, we have

$$\|\hat{\Omega}^{RE} - \tau^R \otimes \sigma^E\|_1 \leq 2\sqrt{2\sqrt{4\varepsilon}} := \kappa. \quad (25)$$

From (23) it follows that $\Omega^{RE} = \hat{\Omega}^{RE}$, since the decoding map does not act on the systems R and E . This fact, together with (25) implies that $\tau^R \otimes \sigma^E \in \mathcal{B}^{2\sqrt{\kappa}}(\Omega^{RE})$ and $\tau^R \in \mathcal{B}^{2\sqrt{\kappa}}(\Omega^R)$.

Let us set $A \equiv RB_1$ in the state $\Omega^{RB_1 BE}$ defined in (24), and let $\varepsilon' \geq 0$ and $\varepsilon'' = 2\sqrt{\kappa}$. Then

$$\begin{aligned} & -H_{\max}^{\varepsilon+2\varepsilon'+\varepsilon''}(A|B)_\Omega \\ &= H_{\min}^{\varepsilon+2\varepsilon'+\varepsilon''}(RB_1|E)_\Omega \\ &\geq H_{\min}^{\varepsilon''}(R|E)_\Omega + H_{\min}^{\varepsilon'}(B_1|RE)_\Omega - \log \frac{2}{\varepsilon^2} \\ &\geq H_{\min}(R)_\tau + H_{\min}^{\varepsilon'}(B_1|RE)_\Omega - \log \frac{2}{\varepsilon^2} \\ &\geq H_{\min}(R)_\tau + H_{\min}^{\varepsilon'}(B_1|REB)_\Omega - \log \frac{2}{\varepsilon^2} \\ &= H_{\min}(R)_\tau - H_{\max}^{\varepsilon'}(B_1)_\Omega - \log \frac{2}{\varepsilon^2} \\ &\geq \log |A_0| - \log |A_1| - \log \frac{2}{\varepsilon^2}. \end{aligned} \quad (26)$$

The first equality holds because of the duality relation (9) between the conditional smoothed min- and max- entropies. The first inequality follows from the chain rule for smoothed min-entropies (Lemma 19). The second inequality follows from Lemma 20 of Appendix A and $\tau^R \otimes \sigma^E \in \mathcal{B}^{2\sqrt{\kappa}}(\Omega^{RE})$, whereas the third inequality follows from Lemma 21 of Appendix A. The second equality follows from the duality

²This relation uniquely defines the map $\tilde{\mathcal{E}}$ because it specifies its Choi-Jamiołkowski state.

relation (9) and the fact that Ω^{RB_1BE} is a pure state. The last inequality holds because

$$\begin{aligned}\log |A_0| &= \log |R| = H_{\min}(R)_\tau \\ \log |A_1| &= \log |B_1| \geq H_{\max}^{\varepsilon'}(B_1)_\Omega.\end{aligned}$$

Moreover, for any $\varepsilon \geq 0$,

$$H_{\min}^\varepsilon(A)_\Omega \geq H_{\min}(RB_1)_{\tau \otimes \tau} = \log |A_0| + \log |A_1| \quad (27)$$

since $\Omega^{RB_1} = \tau^R \otimes \tau^{B_1}$. Combining (26) and (27) and choosing $\varepsilon' = \varepsilon$ yields

$$\log |A_0| \leq \frac{1}{2} \left[H_{\min}^\varepsilon(A)_\Omega - H_{\max}^{2\varepsilon+2\sqrt{\kappa}}(A|B)_\Omega \right] + \log \frac{\sqrt{2}}{\varepsilon}.$$

This completes the proof of the upper bound in (18) since we can choose Ω^{RB_1BE} to be the pure state corresponding to the channel output (see (24)) when the optimal isometric encoding is applied.

A. One-shot entanglement-assisted quantum (EAQ) capacity of a quantum channel

Definition 9 (Minimum output fidelity): For any Hilbert space \mathcal{H} , we define the minimum output fidelity of a quantum channel \mathcal{N} , in the presence of an assisting maximally entangled state, as follows:

$$\mathbf{F}_{\min}(\mathcal{N}, \mathcal{H}) := \max_{\tilde{\mathcal{E}}, \tilde{\mathcal{D}}} \min_{|\phi\rangle \in \mathcal{H}} F^2(|\phi\rangle, \tilde{\mathcal{D}} \circ \mathcal{N} \circ \tilde{\mathcal{E}}(|\phi\rangle\langle\phi|)), \quad (28)$$

where $\tilde{\mathcal{E}}, \tilde{\mathcal{D}}$ are CPTP maps.

Definition 10: For any fixed $0 < \varepsilon < 1$, the one-shot entanglement-assisted quantum (EAQ) capacity $Q_{\text{ea},\varepsilon}^{(1)}(\mathcal{N})$ of a quantum channel $\mathcal{N}^{A' \rightarrow B}$ is defined as follows:

$$Q_{\text{ea},\varepsilon}^{(1)}(\mathcal{N}) := \max\{\log |\mathcal{H}| : \mathbf{F}_{\min}(\mathcal{N}, \mathcal{H}) \geq 1 - \varepsilon\}. \quad (29)$$

The following theorem allows us to relate the one-shot entanglement-assisted entanglement transmission capacity $E_{\text{ea},\varepsilon}^{(1)}(\mathcal{N})$ to the one-shot entanglement-assisted quantum (EAQ) capacity $Q_{\text{ea},\varepsilon}^{(1)}(\mathcal{N})$.

Theorem 11: For any fixed $0 < \varepsilon < 1$, for a quantum channel $\mathcal{N}^{A' \rightarrow B}$, and an assisting entanglement resource in the form of a maximally entangled state,

$$E_{\text{ea},\varepsilon}^{(1)}(\mathcal{N}) - 1 \leq Q_{\text{ea},2\varepsilon}^{(1)}(\mathcal{N}) \leq E_{\text{ea},4\varepsilon}^{(1)}(\mathcal{N}). \quad (30)$$

Proof: The proof is given in Appendix C. ■

IV. ONE SHOT ENTANGLEMENT-ASSISTED CLASSICAL CAPACITY OF A QUANTUM CHANNEL

We consider entanglement-assisted classical (EAC) communication through a single use of a noisy quantum channel, in the case in which the assisting resource is given by a maximally entangled state. The scenario is depicted in Fig. 2. The sender (Alice) and the receiver (Bob) initially share a maximally entangled state $|\Phi\rangle^{A_1 B_1}$, where Alice possesses A_1 while Bob has B_1 , and $\mathcal{H}_{A_1} \simeq \mathcal{H}_{B_1}$. The goal is for Alice to transmit classical messages labelled by the elements of the set $\mathcal{K} = \{1, 2, \dots, |\mathcal{K}|\}$ to Bob, through a single use of the quantum channel $\mathcal{N} : \mathcal{B}(\mathcal{H}_{A'}) \rightarrow \mathcal{B}(\mathcal{H}_B)$, with the help of the prior shared entanglement.

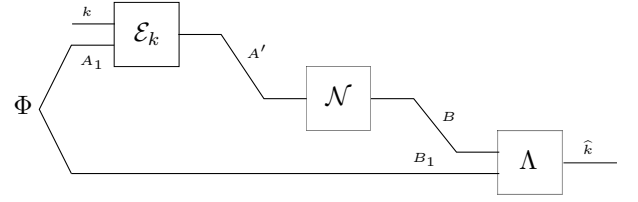


Fig. 2. One-shot entanglement-assisted classical communication. The sender Alice shares a maximally entangled state $|\Phi\rangle^{A_1 B_1}$ with the receiver Bob before the protocol begins. Based on the classical message k , she performs some encoding operation \mathcal{E}_k on her half of the maximally entangled state A_1 before sending it through the quantum channel \mathcal{N} . After receiving the channel output B , Bob performs a POVM Λ on the system $B_1 B$ in his possession, which yields the classical register \hat{K} containing his inference \hat{k} of the message $k \in \mathcal{K}$ sent by Alice.

Without loss of generality, any EAC communication protocol can be assumed to have the following form: Alice encodes her classical messages into states of the system A_1 in her possession. Let the encoding (CPTP) map corresponding to her k^{th} classical message be denoted by $\mathcal{E}_k^{A_1 \rightarrow A'}$, for each $k \in \mathcal{K}$. Alice then sends the system A' through the noisy quantum channel $\mathcal{N}^{A' \rightarrow B}$. After Bob receives the channel output B , he performs a POVM $\Lambda : B_1 B \rightarrow \hat{K}$ on the system $B_1 B$ in his possession, which yields the classical register \hat{K} containing his inference \hat{k} of the message $k \in \mathcal{K}$ sent by Alice.

Definition 12 (One-shot ε -error EAC capacity): Given a quantum channel $\mathcal{N}^{A' \rightarrow B}$ and a real number $0 < \varepsilon < 1$, the one-shot ε -error entanglement-assisted classical capacity of \mathcal{N} is defined as follows:

$$C_{\text{ea},\varepsilon}^{(1)}(\mathcal{N}) := \max\{\log |\mathcal{K}| : \forall k \in \mathcal{K}, \Pr[k \neq \hat{k}] \leq \varepsilon\} \quad (31)$$

where the maximization is over all possible encoding operations and POVMs.

The following theorem gives bounds on the one-shot ε -error EAC capacity of a quantum channel.

Theorem 13: For any fixed $0 < \varepsilon < 1$, the one-shot ε -error entanglement-assisted classical capacity of a noisy quantum channel $\mathcal{N}^{A' \rightarrow B}$, in the case in which the assisting resource is a maximally entangled state, satisfies the following bounds

$$\begin{aligned} \max_{\phi^{A'}} \left[H_{\min}^{\varepsilon''}(A)_\psi - H_{\max}^{\varepsilon''}(A|B)_\psi \right] + 4 \log \varepsilon'' - 2 \\ \leq C_{\text{ea},\varepsilon}^{(1)}(\mathcal{N}) \leq \\ \max_{\phi^{A'}} \left[H_{\min}^{4\varepsilon}(A)_\psi - H_{\max}^{8\varepsilon+2\sqrt{\kappa'}}(A|B)_\psi \right] + \log \frac{1}{2\sqrt{2\varepsilon}}, \end{aligned} \quad (32)$$

where the maximization is over all possible input states, $\phi^{A'}$ to the channel, $\kappa' = 2\sqrt{8\sqrt{\varepsilon}}$ and $\varepsilon'' > 0$ is such that $\varepsilon^2 = \sqrt{2\sqrt{27\varepsilon''} + 27\varepsilon''}$. In the above, ψ^{ABE} is the pure state defined in (19).

A. Achievability

The lower bound in (32) is obtained by employing the one-shot version of the entanglement-assisted quantum communication (or “father”) protocol.

From Theorem 8 and Theorem 11, it follows that the one-shot ε -error entanglement-assisted quantum communication protocol for a quantum channel \mathcal{N} that consumes $e_\varepsilon^{(1)}$ ebits

of entanglement and transmits $q_\varepsilon^{(1)}$ qubits can be expressed in terms of the following one-shot resource inequality:

$$\langle \mathcal{N} \rangle + e_\varepsilon^{(1)}[qq] \geq_\varepsilon q_\varepsilon^{(1)}[q \rightarrow q]. \quad (33)$$

Here $[q \rightarrow q]$ represents one qubit of quantum communication from Alice (the sender) to Bob (the receiver); $[qq]$ represents an ebit shared between Alice and Bob, and the notation \geq_ε is used to emphasize that the error in achieving the goal of the protocol is at most ε . In the above, $e_\varepsilon^{(1)}$ and $q_\varepsilon^{(1)}$, given below, respectively, follow from (80) and (81):

$$e_\varepsilon^{(1)} = \frac{1}{2} \left[H_{\min}^{\varepsilon'}(A)_\psi + H_{\max}^{\varepsilon'}(A|B)_\psi \right] \quad (34)$$

$$q_\varepsilon^{(1)} = \frac{1}{2} \left[H_{\min}^{\varepsilon'}(A)_\psi - H_{\max}^{\varepsilon'}(A|B)_\psi \right] + 2 \log \varepsilon' - 1, \quad (35)$$

where $\varepsilon' > 0$ is such that $\varepsilon = \sqrt{2\sqrt{27\varepsilon'} + 27\varepsilon'}$, and ψ^{ABE} is defined in (19).

The resource inequality (33) readily yields a resource inequality for one-shot EAC communication through a noisy quantum channel, which in turn can be used to obtain a lower bound on the one-shot EAC capacity. This can be seen as follows. Combining (33) with the resource inequality for superdense coding:

$$[qq] + [q \rightarrow q] \geq 2[c \rightarrow c],$$

yields the following resource inequality for one-shot EAC communication through the noisy channel $\mathcal{N} \equiv \mathcal{N}^{A' \rightarrow B}$:

$$\begin{aligned} \langle \mathcal{N} \rangle + q_\varepsilon^{(1)}[qq] + e_\varepsilon^{(1)}[qq] &\geq_\varepsilon q_\varepsilon^{(1)}[q \rightarrow q] + q_\varepsilon^{(1)}[qq] \\ &\geq_{\sqrt{\varepsilon}} 2q_\varepsilon^{(1)}[c \rightarrow c]. \\ \Rightarrow \langle \mathcal{N} \rangle + (q_\varepsilon^{(1)} + e_\varepsilon^{(1)})[qq] &\geq_{\sqrt{\varepsilon}} 2q_\varepsilon^{(1)}[c \rightarrow c]. \end{aligned} \quad (36)$$

Replacing ε by ε^2 in (36) directly yields the following lower bound on the ε -error one-shot EAC capacity³:

$$\begin{aligned} C_{\text{ea},\varepsilon}^{(1)}(\mathcal{N}) &\geq 2q_{\varepsilon^2}^{(1)} \\ &= \left[H_{\min}^{\varepsilon''}(A)_\psi - H_{\max}^{\varepsilon''}(A|B)_\psi \right] + 4 \log \varepsilon'' - 2, \end{aligned} \quad (37)$$

where ε'' is as defined in Theorem 13. Note that (37) reduces to the lower bound in (32) when the optimal input state $\phi^{A'}$ is used.

B. Proof of the Converse

We prove the upper bound in (32) by showing that if it did not hold then one would obtain a contradiction to the upper bound (in (18)) on the one-shot entanglement-assisted quantum capacity of a channel.

Let us assume that

$$C_{\text{ea},\varepsilon}^{(1)}(\mathcal{N}) > \Delta(2\varepsilon, \mathcal{N}), \quad (38)$$

³The necessity of replacing ε by ε^2 arises from the different fidelity criteria used in defining the one-shot entanglement assisted quantum and classical capacities (see (28) and (31))

where

$$\begin{aligned} \Delta(2\varepsilon, \mathcal{N}) &:= \\ \max_{\phi^{A'}} \left[H_{\min}^{4\varepsilon}(A)_\psi - H_{\max}^{8\varepsilon+2\sqrt{\kappa'}}(A|B)_\psi \right] &+ \log \frac{1}{2\sqrt{2\varepsilon}}, \end{aligned} \quad (39)$$

with $\kappa' := 2\sqrt{8\sqrt{\varepsilon}}$, ψ^{ABE} being given by (19), and the maximization being over all possible input states to the channel⁴. This is equivalent to the assumption that more than $\Delta(2\varepsilon, \mathcal{N})$ bits of classical information can be communicated through a single use of \mathcal{N} with an error $\leq \varepsilon$, in the presence of an entanglement resource in the form of a maximally entangled state.

Now since unlimited entanglement is available for the protocol, we infer that by quantum teleportation more than $\Delta(2\varepsilon, \mathcal{N})/2$ qubits can be transmitted over a single use of \mathcal{N} with an error $\leq 2\varepsilon$. Then from the definition (29) of the one-shot 2ε -error entanglement assisted quantum capacity $Q_{\text{ea},2\varepsilon}^{(1)}(\mathcal{N})$ of the channel \mathcal{N} , it follows that $Q_{\text{ea},2\varepsilon}^{(1)}(\mathcal{N}) > \Delta(2\varepsilon, \mathcal{N})/2$. However, this contradicts the upper bound to $Q_{\text{ea},2\varepsilon}^{(1)}(\mathcal{N})$ as obtained from (30) and (18).

V. ENTANGLEMENT-ASSISTED CLASSICAL AND QUANTUM CAPACITIES FOR MULTIPLE USES OF A MEMORYLESS CHANNEL

A. Entanglement-assisted classical capacity for multiple uses of a memoryless channel

Definition 14: We define the entanglement-assisted classical capacity in the asymptotic memoryless scenario as follows:

$$C_{\text{ea}}^\infty(\mathcal{N}) := \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} C_{\text{ea},\varepsilon}^{(1)}(\mathcal{N}^{\otimes n})$$

where $C_{\text{ea},\varepsilon}^{(1)}(\mathcal{N}^{\otimes n})$ denotes the one-shot ε -error EAC capacity for n independent uses of the channel \mathcal{N} .

Next we show how the known achievable rate for EAC communication in the asymptotic, memoryless scenario can be recovered from Theorem 13. We also prove that this rate is indeed optimal [9], [10].

Theorem 15: [9], [10] The entanglement-assisted classical capacity in the asymptotic memoryless scenario is given by the following:

$$C_{\text{ea}}^\infty(\mathcal{N}) = \max_{\phi^{A'} \in \mathcal{D}(\mathcal{H}_{A'})} I(A : B)_\psi \quad (40)$$

where the maximization is over all possible input states to the channel \mathcal{N} , ψ^{ABE} is defined in (19), and $I(A : B)_\psi$ denotes the mutual information of the state $\psi^{AB} := \text{Tr}_E \psi^{ABE}$.

Proof: First we prove that

$$C_{\text{ea}}^\infty(\mathcal{N}) \geq \max_{\phi^{A'} \in \mathcal{D}(\mathcal{H}_{A'})} I(A : B)_\psi. \quad (41)$$

⁴The factor of 2 in front of ε arises from the different fidelity criteria used in defining the one-shot entanglement assisted quantum and classical capacities (see (28) and (31)), and the relation (3) between the fidelity and the trace distance.

From the lower bound in Theorem 13, we have

$$C_{\text{ea}}^\infty(\mathcal{N}) \geq \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \left(\max_{\phi^{A'n} \in \mathcal{D}(\mathcal{H}_{A'}^{\otimes n})} [H_{\min}^{\varepsilon''}(A^n)_{\psi_n} - H_{\max}^{\varepsilon''}(A^n|B^n)_{\psi_n}] + 4 \log \varepsilon'' - 2 \right)$$

where ψ_n is defined as

$$\psi_n \equiv \psi^{A^n B^n} := (\text{id}^{A^n} \otimes \mathcal{N}^{\otimes n})(\phi^{A^n A'^n}), \quad (42)$$

where $\phi^{A^n A'^n}$ denotes a purification of the input state $\phi^{A'^n}$, and ε'' is as defined in Theorem 13. By restricting the maximization in the above inequality to the set of input states of the form $(\phi^{A'})^{\otimes n}$

$$C_{\text{ea}}^\infty(\mathcal{N}) \geq \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \left(\max_{\phi^{A'} \in \mathcal{D}(\mathcal{H}_{A'})} [H_{\min}^{\varepsilon''}(A^n)_{\psi^{\otimes n}} - H_{\max}^{\varepsilon''}(A^n|B^n)_{\psi^{\otimes n}}] + 4 \log \varepsilon'' - 2 \right) \quad (43)$$

where ψ is defined through (19). Let $\hat{\psi}^{AB}$ be the state such that

$$I(A : B)_{\hat{\psi}} = \max_{\phi^{A'} \in \mathcal{D}(\mathcal{H}_{A'})} I(A : B)_{\psi}.$$

Further restricting to the state $\hat{\psi}^{AB}$, we can obtain the following from (43)

$$C_{\text{ea}}^\infty(\mathcal{N}) \geq \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} [H_{\min}^{\varepsilon''}(A^n)_{\hat{\psi}^{\otimes n}} - H_{\max}^{\varepsilon''}(A^n|B^n)_{\hat{\psi}^{\otimes n}} + 4 \log \varepsilon'']. \quad (44)$$

Then from the superadditivity of the limit inferior and the fact that the limits on the right-hand side of the above equation exist [26], we obtain

$$C_{\text{ea}}^\infty(\mathcal{N}) \geq \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} H_{\min}^{\varepsilon''}(A^n)_{\hat{\psi}^{\otimes n}} - \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} H_{\max}^{\varepsilon''}(A^n|B^n)_{\hat{\psi}^{\otimes n}}. \quad (45)$$

Finally, by using Lemma 22, we obtain the desired bound (41):

$$\begin{aligned} C_{\text{ea}}^\infty(\mathcal{N}) &\geq H(A)_{\hat{\psi}} - H(A|B)_{\hat{\psi}} \\ &= I(A : B)_{\hat{\psi}}. \end{aligned}$$

Next we prove that

$$C_{\text{ea}}^\infty(\mathcal{N}) \leq \max_{\phi^{A'} \in \mathcal{D}(\mathcal{H}_{A'})} I(A : B)_{\psi}. \quad (46)$$

From the upper bound in Theorem 13, we have

$$C_{\text{ea}}^\infty(\mathcal{N}) \leq \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \left(\max_{\phi^{A'n} \in \mathcal{D}(\mathcal{H}_{A'}^{\otimes n})} [H_{\min}^{4\varepsilon}(A^n)_{\psi_n} - H_{\max}^{8\varepsilon+2\sqrt{\kappa'}}(A^n|B^n)_{\psi_n}] + \log \frac{1}{2\sqrt{2\varepsilon}} \right).$$

Using Lemma 23, we obtain

$$C_{\text{ea}}^\infty(\mathcal{N}) \leq \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \left(\max_{\phi^{A'n} \in \mathcal{D}(\mathcal{H}_{A'}^{\otimes n})} [H(A^n)_{\psi_n} - H(A^n|B^n)_{\psi_n}] + f(\varepsilon, n) \right), \quad (47)$$

where

$$f(\varepsilon, n) := \log \frac{1}{2\sqrt{2\varepsilon}} + 256\varepsilon \log |A^n| + 4h(48\varepsilon).$$

In the above, we have used the notation $h(\eta)$ to denote the binary entropy for any $0 \leq \eta \leq 1$. The fact that the limit inferior is upper bounded by the limit superior and the latter is subadditive, and the fact that

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} f(\varepsilon, n) = 0,$$

imply that

$$C_{\text{ea}}^\infty(\mathcal{N}) \leq \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \left(\max_{\phi^{A'n} \in \mathcal{D}(\mathcal{H}_{A'}^{\otimes n})} [H(A^n)_{\psi_n} - H(A^n|B^n)_{\psi_n}] \right). \quad (48)$$

The above equation reduces to

$$C_{\text{ea}}^\infty(\mathcal{N}) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \left(\max_{\phi^{A'n} \in \mathcal{D}(\mathcal{H}_{A'}^{\otimes n})} [I(A^n : B^n)_{\psi_n}] \right),$$

since the limit in (48) exists, and $H(A^n)_{\psi_n} - H(A^n|B^n)_{\psi_n} = I(A^n : B^n)_{\psi_n}$. Using eq.(3.24) of [31] we infer that $I(A^n : B^n)_{\psi_n}$ is subadditive since the channel from whose action the state ψ^n (defined by (42)), results is memoryless (i.e., $\mathcal{N}^{\otimes n}$). This yields

$$C_{\text{ea}}^\infty(\mathcal{N}) \leq \max_{\phi^{A'} \in \mathcal{D}(\mathcal{H}_{A'})} I(A : B)_{\psi}. \quad (49)$$

■

B. Entanglement-assisted quantum capacity for multiple uses of a memoryless channel

Definition 16: We define the entanglement-assisted quantum capacity of a quantum channel \mathcal{N} in the asymptotic memoryless scenario as follows:

$$Q_{\text{ea}}^\infty(\mathcal{N}) := \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} Q_{\text{ea}, \varepsilon}^{(1)}(\mathcal{N}^{\otimes n})$$

where $Q_{\text{ea}, \varepsilon}^{(1)}(\mathcal{N}^{\otimes n})$ denotes the one-shot ε -error entanglement-assisted quantum capacity for n independent uses of the channel \mathcal{N} .

Theorem 17: [13] The entanglement-assisted quantum capacity of a quantum channel \mathcal{N} in the asymptotic, memoryless scenario is given by

$$Q_{\text{ea}}^\infty(\mathcal{N}) = \max_{\phi^{A'} \in \mathcal{D}(\mathcal{H}_{A'})} \frac{1}{2} I(A : B)_{\psi} \quad (50)$$

where ψ^{ABE} is defined in (19), and the maximisation is over all possible input states to the channel.

Proof: The proof is exactly analogous to the proof of Theorem 15. ■

VI. CONCLUSIONS

We established upper and lower bounds on the one-shot entanglement-assisted quantum and classical capacities of a quantum channel, and proved that these bounds converge to the known single-letter formulas for the corresponding capacities in the asymptotic, memoryless scenario. The bounds in the one-shot case are given in terms of the difference of two smoothed entropic quantities. This quantity serves as a one-shot analogue of mutual information, since it reduces to the mutual information between the output of a channel and a system purifying its input in the asymptotic, memoryless scenario. Note that it is similar in form to the expression characterizing the one-shot capacity of a c-q channel as obtained in [16].

There are some other quantities in the existing literature on one-shot quantum information theory which could also be considered to be one-shot analogues of mutual information, namely, the quantity characterizing the classical capacity of a c-q channel [15] which is defined in terms of the relative Rényi entropy of order zero, and a quantity characterizing the quantum communication cost of a one-shot quantum state splitting protocol [32], which is defined in terms of the max-relative entropy. It would be interesting to investigate how the quantity arising in this paper and these different one-shot analogues of the mutual information are related to each other.

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APPENDIX A USEFUL LEMMAS

We make use of the following properties of the min- and max-entropies which were proved in [25], [33]:

Lemma 18: Let $0 < \varepsilon \leq 1$, $\rho^{AB} \in \mathcal{D}_{\leq}(\mathcal{H}_{AB})$, and let $U^{A \rightarrow C}$ and $V^{B \rightarrow D}$ be two isometries with $\omega^{CD} := (U \otimes V)\rho^{AB}(U^\dagger \otimes V^\dagger)$, then

$$\begin{aligned} H_{\min}^\varepsilon(A|B)_\rho &= H_{\min}^\varepsilon(C|D)_\omega \\ H_{\max}^\varepsilon(A|B)_\rho &= H_{\max}^\varepsilon(C|D)_\omega. \end{aligned}$$

Lemma 19 (Chain rule for smoothed min-entropy): [33] Let $\varepsilon > 0$, $\varepsilon', \varepsilon'' \geq 0$ and $\rho^{ABC} \in \mathcal{D}(\mathcal{H}_{ABC})$. Then

$$H_{\min}^{\varepsilon+2\varepsilon'+\varepsilon''}(A|B|C)_\rho \geq H_{\min}^{\varepsilon'}(A|BC)_\rho + H_{\min}^{\varepsilon''}(B|C)_\rho - \log \frac{2}{\varepsilon^2}.$$

Lemma 20: Let $\varepsilon > 0$ and $\delta \geq 0$. Then

$$H_{\min}^{\delta+\varepsilon}(A|B)_\rho \geq H_{\min}^\delta(A)_\sigma \quad (51)$$

whenever $\rho^{AB} \in \mathcal{B}^\varepsilon(\sigma^A \otimes \varrho^B)$ for some $\sigma^A \in \mathcal{D}(\mathcal{H}_A)$ and $\varrho^B \in \mathcal{D}(\mathcal{H}_B)$.

Proof: For $\rho \in \mathcal{D}(\mathcal{H})$ and $\sigma \in \mathcal{D}_{\leq}(\mathcal{H})$, define

$$C(\rho, \sigma) := \sqrt{1 - F^2(\rho, \sigma)}. \quad (52)$$

This quantity $C(\rho, \sigma)$ was introduced in [35] and proved to be a metric. It is monotonic under any CPTP map \mathcal{E} , i.e.,

$$C(\rho, \sigma) \geq C(\mathcal{E}(\rho), \mathcal{E}(\sigma)). \quad (53)$$

Moreover, if $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, then

$$C(\rho, \sigma) \leq \sqrt{\|\rho - \sigma\|_1}. \quad (54)$$

This follows by noting that $C(\rho, \sigma)$ is a special case of the purified distance $P(\rho, \sigma)$ (introduced in [25]), which satisfies these properties.

For any $\bar{\sigma}^A \in \mathcal{B}^\delta(\sigma^A)$, we have

$$\begin{aligned} C(\bar{\sigma}^A \otimes \varrho^B, \rho^{AB}) &\leq C(\bar{\sigma}^A \otimes \varrho^B, \sigma^A \otimes \varrho^B) + C(\sigma^A \otimes \varrho^B, \rho^{AB}) \\ &\leq \delta + \varepsilon \end{aligned} \quad (55)$$

In other words, $\forall \bar{\sigma}^A \in \mathcal{B}^\delta(\sigma^A)$, the following is true:

$$\bar{\sigma}^A \otimes \varrho^B \in \mathcal{B}^{\delta+\varepsilon}(\rho^{AB}).$$

We then have

$$\begin{aligned} &H_{\min}^{\delta+\varepsilon}(A|B)_\rho \\ &= \max_{\bar{\rho}^{AB} \in \mathcal{B}^{\delta+\varepsilon}(\rho^{AB})} H_{\min}(A|B)_{\bar{\rho}} \\ &\geq \max_{\bar{\sigma}^A \in \mathcal{B}^\delta(\sigma^A)} H_{\min}(A|B)_{\bar{\sigma}^A \otimes \varrho^B} \\ &= \max_{\bar{\sigma}^A \in \mathcal{B}^\delta(\sigma^A)} \max_{\varphi^B \in \mathcal{D}(\mathcal{H}_B)} [-D_{\max}(\bar{\sigma}^A \otimes \varrho^B \| \mathbb{I}_A \otimes \varphi^B)] \\ &\geq \max_{\bar{\sigma}^A \in \mathcal{B}^\delta(\sigma^A)} [-D_{\max}(\bar{\sigma}^A \otimes \varrho^B \| \mathbb{I}_A \otimes \varrho^B)] \\ &= \max_{\bar{\sigma}^A \in \mathcal{B}^\delta(\sigma^A)} H_{\min}(A)_{\bar{\sigma}^A} \\ &= H_{\min}^\delta(A)_\sigma. \end{aligned} \quad (56)$$

The first inequality follows because $\bar{\sigma}^A \otimes \varrho^B \in \mathcal{B}^{\delta+\varepsilon}(\rho^{AB})$ for any $\bar{\sigma}^A \in \mathcal{B}^\delta(\sigma^A)$. The second inequality follows because we choose a particular $\varphi^B = \varrho^B$. ■

Lemma 21: [25] Let $0 < \varepsilon \leq 1$ and $\rho^{ABC} \in \mathcal{D}_{\leq}(\mathcal{H}_{ABC})$. Then

$$\begin{aligned} H_{\min}^\varepsilon(A|BC)_\rho &\leq H_{\min}^\varepsilon(A|B)_\rho \\ H_{\max}^\varepsilon(A|BC)_\rho &\leq H_{\max}^\varepsilon(A|B)_\rho. \end{aligned}$$

The following identity is given in Theorem 1 of [26]:

Lemma 22: $\forall \rho^{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$,

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} H_{\min}^\varepsilon(A|B)_{\rho^{\otimes n}} = H(A|B)_\rho \quad (57)$$

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} H_{\max}^\varepsilon(A|B)_{\rho^{\otimes n}} = H(A|B)_\rho. \quad (58)$$

The following lemma is given in Lemma 5 of [16].

Lemma 23: For any $1 > \varepsilon > 0$, and $\rho^{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, we have

$$H_{\min}^\varepsilon(A|B)_\rho \leq H(A|B)_\rho + 8\varepsilon \log |A| + 2h(2\varepsilon) \quad (59)$$

$$H_{\max}^\varepsilon(A|B)_\rho \geq H(A|B)_\rho - 8\varepsilon \log |A| - 2h(2\varepsilon) \quad (60)$$

where $h(\varepsilon) = -\varepsilon \log \varepsilon - (1 - \varepsilon) \log(1 - \varepsilon)$.

The following lemma is from [33].

Lemma 24: For $1 > \varepsilon > 0$, $\rho^{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $\rho^{A'B'} \in \mathcal{D}(\mathcal{H}_{A'} \otimes \mathcal{H}_{B'})$, we have

$$H_{\min}^{2\varepsilon}(AA'|BB')_{\rho \otimes \rho'} \geq H_{\min}^\varepsilon(A|B)_\rho + H_{\min}^\varepsilon(A'|B')_{\rho'}. \quad (61)$$

APPENDIX B
PROOF OF THE LOWER BOUND IN (18)

We need the following one-shot decoupling lemma [34].

Theorem 25 (One-shot decoupling): Fix $\varepsilon > 0$, and a completely positive and trace-preserving map $\mathcal{N}^{A'' \rightarrow B}$ with Stinespring extension $U_{\mathcal{N}}^{A'' \rightarrow B\tilde{E}}$ and a complementary channel $\mathcal{N}_c^{A'' \rightarrow \tilde{E}}$. Let $\phi^{\tilde{A}\tilde{B}\tilde{R}}$ be a pure state shared among Alice, Bob, and a reference, and let $\Omega^{AB\tilde{E}} := (\text{id}^A \otimes U_{\mathcal{N}}^{A'' \rightarrow B\tilde{E}})\sigma^{AA''}$, where $\sigma \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_{A''})$ is any pure state. Then there exists an encoding partial isometry $V^{A \rightarrow A''}$ and a decoding map $\mathcal{D}^{B\tilde{B} \rightarrow \tilde{A}\tilde{B}}$ such that

$$\|\mathcal{N}_c^{A'' \rightarrow \tilde{E}}(V(\phi^{\tilde{A}\tilde{B}\tilde{R}})V^\dagger) - \Omega^{\tilde{E}} \otimes \phi^{\tilde{R}}\|_1 \leq 2\sqrt{\delta_1} + \delta_2 \quad (62)$$

and

$$\|(\mathcal{D} \circ \mathcal{N})(V(\phi^{\tilde{A}\tilde{B}\tilde{R}})V^\dagger) - \phi^{\tilde{A}\tilde{B}\tilde{R}}\|_1 \leq 2\sqrt{2\sqrt{\delta_1} + \delta_2} \quad (63)$$

where

$$\delta_1 = 3 \times 2^{\frac{1}{2}} [H_{\max}^{\varepsilon}(\tilde{A})_{\phi} - H_{\min}^{\varepsilon}(A)_{\Omega}] + 24\varepsilon \quad (64)$$

$$\delta_2 = 3 \times 2^{-\frac{1}{2}} [H_{\min}^{\varepsilon}(A|\tilde{E})_{\Omega} + H_{\min}^{\varepsilon}(\tilde{A}|\tilde{R})_{\psi}] + 24\varepsilon. \quad (65)$$

Proof:

Identify the state $\phi^{\tilde{A}\tilde{B}\tilde{R}}$ in Theorem 25 with the state $\Phi^{RA_0} \otimes \Phi^{A_1B_1}$, where $\tilde{A} \equiv A_0A_1$, $\tilde{B} \equiv B_1$ and $\tilde{R} \equiv R$. Identify the state $\Omega^{AB\tilde{E}}$ in Theorem 25 with

$$\Omega^{AB\tilde{E}} := (\text{id}^A \otimes U_{\mathcal{N}}^{A' \rightarrow BEE_1})(\varphi^{AA'}). \quad (66)$$

where $U_{\mathcal{N}}^{A' \rightarrow BE}$ is a Stinespring extension of the map $\mathcal{N}^{A' \rightarrow B}$ and $\varphi^{AA'}$ is some pure state.

Theorem 25 states that a partial isometry $V^{A_0A_1 \rightarrow A''}$ and a decoding map $\mathcal{D}^{B_1B \rightarrow \hat{A}_0\hat{A}_1\hat{B}_1}$ exist such that

$$\|\mathcal{N}_c^{A' \rightarrow E}(V(\Phi^{A_0R} \otimes \tau^{A_1})V^\dagger) - \Omega^E \otimes \tau^R\|_1 \leq 2\sqrt{\delta_1} + \delta_2 \quad (67)$$

and

$$\|\hat{\Omega}^{\hat{A}_0\hat{A}_1\hat{B}_1R} - \Phi^{A_0R} \otimes \Phi^{A_1B_1}\|_1 \leq 2\sqrt{2\sqrt{\delta_1} + \delta_2} \quad (68)$$

where $\mathcal{N}_c^{A' \rightarrow E}$ is the complementary channel induced by \mathcal{N} , the state $\hat{\Omega}^{\hat{A}_0\hat{A}_1\hat{B}_1R}$ is given by

$$\hat{\Omega}^{\hat{A}_0\hat{A}_1\hat{B}_1R} = (\text{id}^R \otimes \mathcal{D} \circ \tilde{\mathcal{N}} \circ V)(\Phi^{A_0R} \otimes \Phi^{A_1B_1}),$$

and

$$\delta_1 = 3 \times 2^{\frac{1}{2}} [H_{\max}^{\varepsilon}(A_0A_1)_{\tau \otimes \tau} - H_{\min}^{\varepsilon}(A)_{\Omega}] + 24\varepsilon \quad (69)$$

$$\delta_2 = 3 \times 2^{-\frac{1}{2}} [H_{\min}^{\varepsilon}(A|E)_{\Omega} + H_{\min}^{\varepsilon}(A_0A_1|R)_{\Phi \otimes \tau}] + 24\varepsilon. \quad (70)$$

If

$$H_{\max}^{\varepsilon}(A_0A_1)_{\tau \otimes \tau} - H_{\min}^{\varepsilon}(A)_{\Omega} \leq 2\log \varepsilon \quad (71)$$

$$H_{\min}^{\varepsilon}(A|E)_{\Omega} + H_{\min}^{\varepsilon}(A_0A_1|R)_{\Phi \otimes \tau} \geq -2\log \varepsilon \quad (72)$$

then

$$\delta_1 \leq 27\varepsilon \quad (73)$$

$$\delta_2 \leq 27\varepsilon. \quad (74)$$

From (11) and the additivity of the max-entropy for tensor-product states, we have

$$\begin{aligned} \text{L.H.S. of (71)} &\leq H_{\max}(A_0)_{\tau} + H_{\max}(A_1)_{\tau} - H_{\min}^{\varepsilon}(A)_{\Omega} \\ &= \log |A_0| + \log |A_1| - H_{\min}^{\varepsilon}(A)_{\Omega}. \end{aligned} \quad (75)$$

Note that (71) (and hence also (73)) is satisfied for the choice

$$\text{R.H.S. of (75)} \leq 2\log \varepsilon. \quad (76)$$

This in turn yields

$$\log |A_0| + \log |A_1| \leq H_{\min}^{\varepsilon}(A)_{\Omega} + 2\log \varepsilon. \quad (77)$$

Similarly, using Lemma 24 of Appendix A, we have

$$\begin{aligned} \text{L.H.S. of (72)} &\geq H_{\min}^{\varepsilon}(A|E)_{\Omega} + H_{\min}^{\varepsilon/2}(A_0|R)_{\Phi} + H_{\min}^{\varepsilon/2}(A_1)_{\tau} \\ &\geq -H_{\max}^{\varepsilon}(A|B)_{\Omega} - \log |A_0| + \log |A_1|. \end{aligned} \quad (78)$$

Then setting

$$\text{R.H.S. of (78)} \geq -2\log \varepsilon,$$

for which (72) (and hence also (74)) is satisfied, we obtain

$$\log |A_0| - \log |A_1| \leq -H_{\max}^{\varepsilon}(A|B)_{\Omega} + 2\log \varepsilon. \quad (79)$$

We further have

$$F(\hat{\Omega}^{R\hat{A}_0}, \Phi^{RA_0}) \geq 1 - \frac{1}{2} \|\hat{\Omega}^{R\hat{A}_0} - \Phi^{RA_0}\|_1 \geq 1 - \varepsilon'.$$

where $\varepsilon' = \sqrt{2\sqrt{27\varepsilon} + 27\varepsilon}$. This in turn implies that there exists a pair (V, \mathcal{D}) such that

$$F_e(\tau^{A_0}, \tilde{\mathcal{D}} \circ \mathcal{N} \circ \tilde{\mathcal{E}}) \equiv F^2(\hat{\Omega}^{R\hat{A}_0}, \Phi^{RA_0}) \geq 1 - 2\varepsilon'$$

where $\tilde{\mathcal{E}}(\Phi^{A_0R}) = V(\Phi^{A_0R} \otimes \Phi^{A_1B_1})$ and $\tilde{\mathcal{D}} := \text{Tr}_{\hat{A}_1\hat{B}_1} \circ \mathcal{D}$. Finally, from (77) and (79) we infer that the quantum communication gain $\log |A_0|$ and the entanglement cost $\log |A_1|$, as given below, respectively, are achievable:

$$\log |A_0| = \frac{1}{2} [H_{\min}^{\varepsilon}(A)_{\Omega} - H_{\max}^{\varepsilon}(A|B)_{\Omega}] + 2\log \varepsilon \quad (80)$$

$$\log |A_1| = \frac{1}{2} [H_{\min}^{\varepsilon}(A)_{\Omega} + H_{\max}^{\varepsilon}(A|B)_{\Omega}]. \quad (81)$$

■

APPENDIX C
PROOF OF THEOREM 11

We employ Lemmas 26 and 27, given below, in our proof. Lemma 26 is obtained directly from Proposition 4.5 of [36]. Lemma 27 yields a relation between the average fidelity and entanglement fidelity and was proved in [36] (Proposition 4.4):

Lemma 26: For any fixed $0 < \varepsilon < 1$, if $\mathcal{T} : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ is a CPTP map such that

$$F_e(\tau, \mathcal{T}) \geq 1 - \varepsilon,$$

where $\tau \in \mathcal{B}(\mathcal{H})$ denotes the completely mixed state, then there exists a subspace $\mathcal{H}' \subset \mathcal{H}$ whose $\dim \mathcal{H}' = (\dim \mathcal{H})/2$, and a CPTP $\mathcal{T}' : \mathcal{B}(\mathcal{H}') \mapsto \mathcal{B}(\mathcal{H}')$ such that

$$F_{\min}(\mathcal{T}') \geq 1 - 2(1 - F_e(\tau, \mathcal{T})) \geq 1 - 2\varepsilon,$$

with \mathcal{T}' being a restriction of the map \mathcal{T} to the subspace \mathcal{H}' (see Proposition 4.5 of [36] for details).

Lemma 27: If $m = \dim \mathcal{H}$ and $\mathcal{T} : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ is a CPTP map, then

$$F_e(\tau, \mathcal{T}) = \frac{(m+1)F_{av}(\mathcal{T}) - 1}{m}$$

where $\tau \in \mathcal{B}(\mathcal{H})$ denotes the completely mixed state.

We now prove Theorem 11.

Proof: The first inequality in (30) is proved as follows. Let $(\mathcal{E}, \mathcal{D})$ be the optimal encoding and decoding pair that achieves the entanglement-assisted entanglement transmission capacity $E_{ea,\varepsilon}^{(1)}(\mathcal{N})$. We then have

$$F_e(\tau^{A_0}, \mathcal{T}) \geq 1 - \varepsilon,$$

where $\mathcal{T} := \tilde{\mathcal{D}} \circ \mathcal{N} \circ \tilde{\mathcal{E}}$ (see (16) for definitions of $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{E}}$). Lemma 26 implies that there exists a subspace $\mathcal{H}' \subset \mathcal{H}_{A_0}$ with dimension $\dim \mathcal{H}' = |A_0|/2$ and a CPTP map $\mathcal{T}' : \mathcal{B}(\mathcal{H}') \mapsto \mathcal{B}(\mathcal{H}')$ (which is a restriction of the map \mathcal{T} to the subspace \mathcal{H}') such that $F_{\min}(\mathcal{T}') \geq 1 - 2\varepsilon$. This implies that

$$Q_{ea,2\varepsilon}^{(1)}(\mathcal{N}) \geq \log |\mathcal{H}'| = E_{ea,\varepsilon}^{(1)}(\mathcal{N}) - 1.$$

To prove the second inequality in (30), we resort to the average fidelity $F_{av}(\mathcal{T})$ defined in (6). The following relation

$$F_{\min}(\mathcal{T}) \leq F_{av}(\mathcal{T})$$

is trivial from their definitions. Suppose now that $(\mathcal{E}, \mathcal{D})$ is the optimal encoding and decoding pair that achieves the entanglement-assisted quantum capacity $Q_{ea,\varepsilon'}^{(1)}(\mathcal{N})$. We then have

$$F_{\min}(\mathcal{T}) \geq 1 - \varepsilon',$$

where $\mathcal{T} : \mathcal{D} \circ \mathcal{N} \circ \tilde{\mathcal{E}}$. Finally, Lemma 27 ensures that the same encoding and decoding pair $(\mathcal{E}, \mathcal{D})$ will give

$$\begin{aligned} F_e(\tau^{A_0}, \mathcal{T}) &= \frac{(m+1)F_{av}(\mathcal{T}) - 1}{m} \\ &\geq \frac{(m+1)(1 - \varepsilon') - 1}{m} \\ &= 1 - \frac{m+1}{m}\varepsilon' \\ &\geq 1 - 2\varepsilon'. \end{aligned} \quad (82)$$

Therefore, $E_{ea,2\varepsilon'}^{(1)}(\mathcal{N}) \geq Q_{ea,\varepsilon'}^{(1)}(\mathcal{N})$. ■

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